THE UNIVERSITY OF ALBERTA

DEGREE OF APPROXIMATION

BY RATIONAL FUNCTIONS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

IN

MATHEMATICS

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1973

ABSTRACT

The purpose of this thesis is to study the degree of approximation by rational functions.

In Chapter I we discuss the background to the problem of rational approximation.

We present a number of direct and converse theorems about approximation by rational functions in Chapter II, and we observe that rational approximation can be substantially better than polynomial approximation for many classes of functions. The result of D. J. Newman concerning the approximation of the function $|x|$ by a rational function is of fundamental importance in this chapter.

In Chapter III, using methods similar to those of Newman, we obtain the result that there is a rational function which interpolates to $|x|$ on equidistant nodes in $[-1,1]$ with a degree of approximation equal to $O(n^{-1}\log^{-1}n)$. Finally, we extend this result to the case of a piecewise linear continuous function.

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ACKNOWLEDGMENT

The author wishes to express his sincere gratitude to Dr. A. Meir for his guidance and help during the preparation of this thesis.

Appreciation is also expressed to the National Research Council of Canada and the University of Alberta for financial assistance.

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CHAPTER I

INTRODUCTION

Background to the Problem

A problem that arises frequently in both practical and theoretical problems is that of finding a convenient approximation to a given continuous function. Polynomials have been used extensively as the approximating function, and the theory of polynomial approximation is well developed. The basis of this theory is the Weierstrass approximation theorem which guarantees that the maximum deviation of a continuous function defined on a finite interval [a,b] from its best polynomial approximation converges uniformly to zero. The rapidity of this convergence depends on the function and, in particular, upon its smoothness.

A natural extension of polynomial approximation is approximation by means of rational functions, although the theory is far less developed in the latter case. As a quotient of two polynomials of degrees m and n, a rational function $r(x)$ has $m + n + 1$ arbitrary parameters -- the same number as a polynomial of degree $m + n$. However, $r(x)$ has a surprising advantage, since the computation of $r(x)$ for a given argument x does not require $m + n$ additions, $m + n - 1$ multiplications and one division as we might first expect. By writing $r(x)$ in its continued fraction form, the total number of multiplications and divisions is significantly reduced to m or n. Many methods for computing rational approximations presently exist. A discussion of many of these methods may

be found in [5] and [24], for example.

In many instances we shall see that rational approximations have superior convergence properties over polynomial approximations. Since a rational function has singularities at the point where the denominator vanishes, there is the possibility of approximating a function which has the same singularities. In addition, rational approximation may be advantageous for functions with singularities other than poles. Thus rational approximation may succeed where polynomial approximation may fail completely. Another important fact is the possibility of approximation to functions over an infinite interval.

In the proof of the Weierstrass theorem an essential role is played by the condition that the interval [a,b] is finite. In particular, a continuous function defined on [a,b] is always bounded and uniformly continuous. This does not apply to functions defined on an infinite interval since, in such a case, functions may be unbounded, or if bounded may not be uniformly continuous. However, no functions except polynomials can be the limit of a uniformly converging sequence of polynomials on an infinite interval. The use of rational functions as approximating functions partially solves this problem. If the function $f(x)$ is continuous on $(-\infty,\infty)$ and the limits

$$\lim_{x\to-\infty} f(x) \quad , \quad \lim_{x\to\infty} f(x)$$

exist and are finite, then there exists a sequence of rational functions which approximate $f(x)$ arbitrarily closely. Conversely, if a sequence of rational functions converges uniformly on $(-\infty,\infty)$, then

the limit function $f(x)$ is continuous and bounded on $(-\infty,\infty)$ and has unique limits as x tends $\pm\infty$.

The earliest discussion concerning the approximation of a continuous function of a real variable by means of rational functions is due to P. L. Chebyshev [3] who formulated the problem of best approximation in 1859. In the works of Chebyshev [3] and E. I. Zolotarev [36] a number of fundamental results in this direction were obtained. These results included the characterization of the rational function of best approximation and precise expressions for the best approximation by rational functions of certain functions such as $f(x) = \operatorname{sgn} x$.

With regards to the approximation by means of rational functions to an analytic function of a complex variable, the fundamental result is due to C. Runge [22]. It is interesting to note that Runge showed the possibility of rational approximation in 1885, the same year that Weierstrass' famous theorem on polynomial approximation was first published. Substantial progress since this time has been made by J. L. Walsh [33,35].

Throughout this work we shall limit our discussion to the formulation of the fundamental results for functions defined on the real line. We shall emphasize the properties of the rational function of best approximation and compare these results with the corresponding properties of the polynomial of best approximation. However, brief indications of the problems encountered by rational approximation in the complex domain will be given.

We let $R(m,n)$ denote the class of rational functions $r(x)$, defined on the interval $[a,b]$ of the real line, which can be represented in the form

$$r(x) = \frac{p_m(x)}{q_n(x)}$$

where $p_m(x)$ and $q_n(x)$ are polynomials having degrees not greater than m and n, respectively. We shall assume that $q_n(x)$ is not the zero polynomial and that $p_m(x)$ and $q_n(x)$ have no zeros in common. By the <u>order</u> of a rational function we shall mean the maximum of the degrees of its numerator and denominator.

Existence, Uniqueness and Characterization

Immediately we face the question of the existence of the best approximation by rational functions to a given continuous function $f(x)$ on an interval $[a,b]$.

We first define the uniform or Chebyshev norm of a continuous function $f(x)$ on $[a,b]$ by

$$\|f\| = \max_{a \le x \le b} |f(x)| \quad .$$

Then it is well known [32] that there exists a rational function $r^*(x)$ in $R(m,n)$ such that

$$\|f - r^*\| \le \|f - r\|$$

for all rational functions $r(x)$ in $R(m,n)$. Moreover, this $r^*(x)$ is uniquely determined.

This rational function $r^*(x)$ is called the rational function of <u>best approximation</u> to $f(x)$ on $[a,b]$ out of the class $R(m,n)$.

We next turn to the characterization of best rational approximations, first formulated by P. L. Chebyshev [3].

A set $\{x_1, x_2, \ldots, x_N\}$ of N distince points satisfying

$$a \le x_1 < x_2 < \cdots < x_{N-1} < x_N \le b$$

is called an <u>alternating set</u> for $f - r$ if

$$|f(x_i) - r(x_i)| = \|f - r\| \quad , \quad 1 \le i \le N$$

and
$$f(x_i) - r(x_i) = -\{f(x_{i+1}) - r(x_{i+1})\} \quad , \quad 1 \le i \le N-1 \ .$$

Then, if the function $f(x)$ is continuous on $[a,b]$, we have the result that

$$r(x) = \frac{p_m(x)}{q_n(x)}$$

is the rational function of best approximation to $f(x)$ out of $\mathcal{R}(m,n)$ if and only if $f - r$ has an alternating set consisting of

$$N = 2 + \max\{n + \partial p, m + \partial q\}$$

points, where ∂p and ∂q denote the degrees of the polynomials $p_m(x)$ and $q_n(x)$, respectively.

It is clear that if

$$r(x) = \frac{p_m(x)}{q_n(x)}$$

is the best rational approximation to $f(x)$ out of $\mathcal{R}(m,n)$, then it is also the best rational approximation to $f(x)$ out of $\mathcal{R}(k,\ell)$ for $\partial p \le k \le m$ and $\partial q \le \ell \le n$. Moreover, if we let

$$d = \min\{m - \partial p, n - \partial q\} \quad ,$$

then $r(x)$ is in $\mathcal{R}(m-d,n-d)$ but <u>not</u> in $\mathcal{R}(m-d-1,n-d-1)$.

Degree of Approximation

Next we define the degree of approximation by rational functions to a continuous function.

If $f(x)$ is a continuous function on $[a,b]$, we define

$$R_n(f;a,b) = \inf_{r_n} \{ \max_{a\le x\le b} |f(x) - r_n(x)| \} .$$

$R_n(f;a,b)$ is called the degree of approximation to $f(x)$ by rational functions of order n. Sometimes we shall denote $R_n(f;a,b)$ simply by $R_n(f)$.

If $r_n(x)$ is the rational function of best approximation to $f(x)$ on $[a,b]$, then it is clear that

$$R_n(f;a,b) = \| f - r_n \| .$$

In addition, we shall denote by $E_n(f;a,b)$ or $E_n(f)$ the degree of approximation of the function $f(x)$ on $[a,b]$ by means of polynomials of degree n.

Since turning from polynomials to rational functions actually extends the class of approximating functions, it is clear that $R_n(f) \le E_n(f)$. Hence the direct theorems of Jackson [19] on the best approximation by polynomials are also valid for the best approximation by rational functions. However, the question arises as to how much smaller can $R_n(f)$ be than $E_n(f)$. Moreover, what are the relationships between the structural properties of a function $f(x)$ and the rate of convergence of the sequence $\{R_n(f)\}$ to zero? It is only recently that a systematic study of these areas has been

undertaken. Previously it had been conjectured that rational approximation is essentially no better than polynomial approximation.

Particular emphasis has been placed on a study of those functions which can be better approximated by rational functions than by polynomials. However, for some classes of functions $R_n(f)$ and $E_n(f)$ are asymptotically the same. For example, H. S. Shapiro [23] has shown that if $f(x)$ has a modulus of continuity $\omega(\delta)$, then $R_n(f;-1,1) = O(\omega(n^{-1}))$, and that there are functions $f(x)$ for which this estimate cannot be improved. In such a case it is not worthwhile to use rational approximation.

On the other hand, for functions with only a few singularities, rational approximation may be very useful. In fact, wide classes of functions have been found for which this is true.

The fundamental result in this direction was obtained by D. J. Newman [21] who constructed a suitable rational function of order n such that its degree of approximation to the function $f(x) = |x|$ is $O(\exp(-c\sqrt{n}))$ throughout $[-1,1]$. The function $|x|$ is important in the theory of approximation and, in particular, is an important "test" function for the degree of approximation of a given class of approximating functions on $[-1,1]$. For example, Lebesgue's proof of Weierstrass' theorem is based on the approximation of $|x|$.
Newman's result may be stated as follows:

(i) there exists a rational function function $r_n^*(x)$ such that

$$\| |x| - r_n^* \| \le 3\, e^{-\sqrt{n}} \;,$$

(ii) for every rational function $r_n(x)$

$$\left\| |x| - r_n \right\| \geq \frac{1}{2} e^{-9\sqrt{n}} .$$

This result is very remarkable since it is well known that if $p_n(x)$ is the polynomial of degree n of best approximation to $|x|$ on $[-1,1]$, then

$$\frac{c_1}{n} \leq \left\| |x| - p_n \right\| \leq \frac{c_2}{n}$$

where c_1 and c_2 are positive constants.

Newman's methods and ideas have been extended to certain classes of functions which are better approximated by rational functions than by polynomials. The first systematic analysis in this direction was undertaken by P. Szüsz and P. Turán who investigated piecewise analytic functions [30] and functions with convex higher derivatives satisfying a Lipschitz condition [29]. Further results have now been obtained by G. Freud, J. Szabados, A. A. Gončar and others.

CHAPTER II

APPROXIMATION BY RATIONAL FUNCTIONS

The Fundamental Result

The fundamental result is due to D. J. Newman [21] who showed that rational approximation can be substantially better than polynomial approximation. His idea involved the construction of a special rational function which approximates $|x|$.

Let $\xi = \xi_n = \exp(-n^{-1/2})$ where n is a positive integer, and define

$$p_n(x) = \prod_{k=0}^{n-1}(x + \xi^k) , \qquad r_n(x) = x\left(\frac{p_n(x) - p_n(-x)}{p_n(x) + p_n(-x)}\right) .$$

THEOREM 2.1 [21]. If $n \geq 5$, then

$$\big||x| - r_n(x)\big| \leq 3\, e^{-\sqrt{n}}$$

for $-1 \leq x \leq 1$.

In addition, Newman was able to establish a lower bound for the approximation of $|x|$ by rational functions on [-1,1].

THEOREM 2.2 [21]. For every n there does <u>not</u> exist a rational function $r_n(x)$ of order n such that

$$\big||x| - r_n(x)\big| \leq \frac{1}{2}\, e^{-9\sqrt{n}}$$

for $-1 \leq x \leq 1$.

This theorem shows that the rational function constructed in Theorem 2.1 is not "too far" from being the best approximation to $|x|$.

Since these results of Newman first appeared, A. A. Gončar [16] has shown that a better estimate for the constant c in

$$R_n(|x|;-1,1) = O(\exp(-c\sqrt{n}))$$

can be obtained from the work of E. I. Zolotarev [36] on the best approximation of $\operatorname{sgn} x$ by rational functions of order n on the union of the intervals $[-1,-\varepsilon]$ and $[\varepsilon,1]$, where $0 < \varepsilon < 1$. The estimate Gončar has obtained is

$$e^{-5\sqrt{n}} \le R_n(|x|;-1,1) \le e^{-2\sqrt{n}} .$$

Moreover, A. P. Bulanov [1] has used a complicated modification of Newman's methods to show that

$$e^{-\pi\sqrt{n+1}} \le R_n(|x|;-1,1) \le e^{-\pi(1-\delta(n))\sqrt{n}}$$

where $\delta(n) \le c_1 n^{-c_2}$ and c_1, c_2 are positive constants.

Already Newman [21] has observed that the approximation of $|x|$ on [-1,1] is equivalent to the approximation of $\sqrt{x}$ on [0,1]. For, if a rational function function $r_n(x)$ approximates $\sqrt{x}$, then $r_n(x^2)$ approximates $|x|$. Conversely, if $r_n(x)$ approximates $|x|$, then

$$\frac{r_n(\sqrt{x}) + r_n(-\sqrt{x})}{2}$$

approximates $\sqrt{x}$. Furthermore, he suggested that this degree of approximation $O(\exp(-c\sqrt{n}))$ might also be possible for x^α , where α is any positive non-integer rational number. G. Freud and

J. Szabados [13] showed that Newman's method to approximate $\sqrt{x}$ fails for x^α. However, they were able to show the existence of a rational function having order n for which

$$R_n(x^\alpha;0,1) = O(\exp(-c_\alpha n^{1/3})) ,$$

where $\alpha > 0$ is any real number and c_α is a positive constant depending only on α.

More recently, A. A. Gončar [18] has shown that

$$R_n(x^\alpha;0,1) = O(\exp(-c_\alpha \sqrt{n})) ,$$

where $\alpha > 0$ is any real number and c_α is a positive constant depending only on α. These results can be extended in order to obtain an estimate for $R_n(|x|^\alpha;-1,1)$. Noting that

$$\max_{0\le x\le 1} |x^{\alpha/2} - r_n(x)| = O(\exp(-c_{\alpha/2}\sqrt{n}))$$

implies

$$\max_{0\le x\le 1} |x^\alpha - r_n(x^2)| = \max_{-1\le x\le 1} \left||x|^\alpha - r_n(x^2)\right|$$

$$= O(\exp(-c_{\alpha/2}\sqrt{n}))$$

and that the order of $r_n(x^2)$ is no larger than $2n$, we have

$$R_n(|x|^\alpha;-1,1) = O(\exp(-c_\alpha\sqrt{n}))$$

where c_α is a positive constant depending only on α.

The question arises whether and when is rational approximation better than polynomial approximation. For example, Newman [28] has shown that the function

$$f_\alpha(x) = \sum_{m=1}^{\infty} \frac{T_{m!}(\frac{x}{2})}{(m!)^\alpha} , \quad 0 < \alpha < 1$$

(where $T_k(x) = \cos[k \text{ arc } \cos x]$), which satisfies the Lipschitz condition of order α on $[-1,1]$, cannot be approximated on $[-\frac{1}{2},\frac{1}{2}]$ better than $O(n^{-\alpha}\log^{-1} n)$ using rational functions of order n. Since the degree of approximation by polynomials of degree n for the same function is $O(n^{-\alpha})$, the degree of rational approximation is not a significant improvement in this case.

Rational Approximation of Certain Classes of Functions

H. S. Shapiro [23] has remarked that "one might surmise that the main strength of rational approximation lies in the approximation of functions with special analytic properties". P. Szüsz and P. Turán [28] were the first to show that for a general class of functions the approximation by rational functions of order n is essentially better than that by polynomials of degree n.

THEOREM 2.3 [28]. Let $f(x)$ be a convex function on $[-1,1]$ which, in addition, satisfies the Lipschitz condition

$$|f(y) - f(x)| \le M|y - x|$$

for $-1 \le x, y \le 1$. Then, for all $n \ge 2$, there exists a rational function $r_n(x)$ of order n and a positive constant c (depending only on M) such that

$$|f(x) - r_n(x)| \le c \frac{\log^4 n}{n^2}$$

for $-1 \le x \le 1$.

Theorem 2.3 holds in the interval $[-1+\varepsilon, 1-\varepsilon]$, where $0 < \varepsilon < 1$, without the Lipschitz condition. Moreover, Theorem 2.3 is also valid for functions which are indefinite integrals of functions of bounded variation. For, if we represent such a function $f(x)$ as the difference of two monotonic functions, then $f(x)$ is the difference of two convex functions.

On the other hand, Newman [28] has shown that the function

$$f(x) = -Bx^2 + \sum_{m=1}^{\infty} \frac{T_{m!}(\frac{x}{2})}{m^2(m!)^2}$$

(where B is a suitably large positive constant and $T_k(x) = \cos[k \arccos x]$), satisfies the conditions of Theorem 2.3, but its degree of approximation by rational functions of order n cannot be better than $O(n^{-2}\log^{-1} n)$. This suggests that the upper bound given in Theorem 2.3 is almost the best possible estimate.

Szüsz and Turán [29] then obtained a generalization of their first result.

THEOREM 2.4 [29]. Let $f(x)$ be a function which is $(r-1)$-times differentiable for which $f^{(r-1)}(x)$ is convex on $[-1,1]$ and satisfies

$$|f^{(r-1)}(y) - f^{(r-1)}(x)| \le M|y - x| \qquad (r \ge 1)$$

for $-1 \le x, y \le 1$. Then, for $n \ge 2$, there exists a rational function $r_n(x)$ of order n and a positive constant c (depending only on M) such that

$$|f(x) - r_n(x)| \le c\left(\frac{\log^2 n}{n}\right)^{r+1}$$

for $-1 \le x \le 1$.

If $f(x)$ is a function which satisfies the conditions of Theorem 2.4, then we observe that rational approximation to $f(x)$ is better than polynomial approximation. For, in general, the degree of approximation to $f(x)$ by polynomials of degree n is only $O(n^{-r})$.

Later, Szüsz and Turán [30] exhibited a further class of functions which can be better approximated by rational functions than by polynomials.

DEFINITION 2.1. A function $f(x)$ is <u>piecewise analytic</u> on $[-1,1]$ if there exists a subdivision

$$-1 = x_0 < x_1 < \ldots < x_{m-1} < x_m = 1$$

of $[-1,1]$ such that the restriction $f_k(x)$ of $f(x)$ to $[x_{k-1},x_k]$ is analytic on $[x_{k-1},x_k]$, $1 \le k \le m$, and $f_k(x_k) = f_{k+1}(x_k)$, $1 \le k \le m-1$.

The class of piecewise analytic functions is important in different areas of analysis. If all the functions $f_k(x)$, $1 \le k \le m$, are polynomials, then this class includes the so-called spline functions which play an essential role in the approximation and interpolation of functions.

THEOREM 2.5 [30]. If $f(x)$ is piecewise analytic on $[-1,1]$ and $\|f\| \le 1$, then for $n \ge 2$ there exists a rational function $r_n(x)$ of order n such that

$$|f(x) - r_n(x)| \le e^{-c\sqrt{n}}$$

for $-1 \le x \le 1$, where c is a positive constant (depending only on the function f).

A comparison with Theorem 2.2 suggests that the estimate given here is not too far from being best possible. Moreover, polynomial approximation for this class cannot yield anything better than $O(n^{-1})$.

On the other hand, a "break" in the graph of a function at an interior point of the interval means that the degree of rational approximation can be no better than $O(\exp(-c\sqrt{n}))$. For example, if at a point x_0 in $(-1,1)$ the expansions

$$f(x_0-h) - f(x_0) = a_1h + . . . + a_kh^k + O(h^{k+\delta}) ,$$

$$f(x_0+h) - f(x_0) = b_1h + . . . + b_kh^k + O(h^{k+\delta})$$

are valid for some integer $k > 0$ and the coefficients do not depend on h $(h > 0, \delta > 0)$ and if $a_i \neq b_i$ for at least one i, $1 \leq i \leq k$, then there is a constant $c > 0$ such that we have $R_n(f;-1,1) \geq \exp(-c\sqrt{n}))$ [16].

Further results about rational approximation for wider classes of functions were obtained by G. Freud [11].

THEOREM 2.6 [11]. Let $f(x)$ be a continuous function of bounded variation on $[0,1]$. If for constants $0 < \alpha < 1$ and $M > 0$ we have

$$|f(x+h) - f(x)| \leq M|h|^\alpha$$

$(0 \leq x,\ x+h \leq 1)$, then for $n \geq 2$ there exists a rational function $r_n(x)$ of order n such that

$$|f(x) - r_n(x)| \leq c\ \frac{\log^2 n}{n}$$

for $0 \leq x \leq 1$, where c is a positive constant (depending only on α , M and the total variation of f).

Since the degree of approximation by polynomials of degree n to such an $f(x)$ is only $O(n^{-\alpha})$, this result shows that rational approximation to this class of functions is substantially better.

The estimate given in Theorem 2.6 has also been obtained by A. Abdugapparov, E. P. Doľženko and A. P. Bulanov [2].

The next theorem that we give is also due to G. Freud:

THEOREM 2.7 [11]. Let $f(x)$ be a continuous function of bounded variation on [0,1]. If for constants $\gamma > 0$ and $K > 0$ we have

$$|f(x+h) - f(x)| \le K \log^{-\gamma} \frac{1}{|h|}$$

($0 \le x,\ x+h \le 1$), then for $n \ge 2$ there exists a rational function $r_n(x)$ of order n such that

$$|f(x) - r_n(x)| \le c\, n^{-\frac{\gamma}{2+\gamma}}$$

for $0 \le x \le 1$, where c is a positive constant (depending only on γ , K and the total variation of f).

In this case the corresponding degree of polynomial approximation is only $O(\log^{-\gamma} n)$.

The following definition of the class $V^{(r)}$ of functions is to be found in [11].

DEFINITION 2.2. A function $f(x)$ defined on [a,b] is in the class $V^{(r)}$ $(r \ge 1)$ if $f(x)$ is (r-1)-times continuously differentiable and $f^{(r-1)}(x)$ is an indefinite integral of a function $f^{[r]}(x)$ of bounded variation on [a,b].

P. Szüsz and P. Turán [29] dealt with the class $K^{(r)}$ of functions for which $f^{(r)}(x)$ $(r \geq 0)$ is convex. It is easily seen that each function in $V^{(r+1)}$ can be represented as the difference of two functions in $K^{(r)}$. It follows that the class $K^{(r)}$ of functions will give the same degree of approximation as the class $V^{(r+1)}$, and therefore the next theorem, due to Freud [11] is actually a sharpened result of Theorem 2.3 and Theorem 2.4.

THEOREM 2.8 [11]. Let $f(x)$ be a function defined on $[0,1]$ which is in the class $V^{(r)}$ $(r \geq 1)$. Then for $n \geq 2$ there exists a rational function $r_n(x)$ of order n such that

$$|f(x) - r_n(x)| \leq c \frac{\log^2 n}{n^{r+1}}$$

for $0 \leq x \leq 1$, where c is a positive constant (depending only on the total variation of $f^{[r]}$).

If $f(x)$ is a function which is absolutely continuous and $f'(x)$ is of bounded variation with total variation $V(f')$ on $[0,1]$, then we know by the results of Szüsz and Turán [28] and Freud [11] that there is a positive decreasing sequence $\{\lambda_n\}$ of constants with $\lim_{n\to\infty} \lambda_n = 0$ such that

$$R_n(f;0,1) \leq V(f')\,\lambda_n .$$

Theorem 2.8 gives the best known upper estimate of λ_n as $O\left(\frac{\log^2 n}{n^2}\right)$.

In [9] G. Freud has studied the situation for absolutely continuous functions. Later, he extended these results to differentiable functions [10].

Let $f(x)$ be absolutely continuous on $[0,1]$ and consider its modulus of smoothness $\omega_2(f;\delta)$ defined by

$$\omega_2(f;\delta) = \max_{|h|\leq\delta} |f(x+h) - 2f(x) + f(x-h)| \quad .$$

Furthermore, let

$$\Delta(f';h) = \int_0^{1-h} |f'(t+h) - f'(t)|\, dt \quad .$$

(We note that $\lim_{h\to 0} \Delta(f';h) = 0$.)

THEOREM 2.9 [9]. If the function $f(x)$ is absolutely continuous on $[0,1]$, then for every positive integer k there exists a rational function $r_n(x)$ of order n such that

$$|f(x) - r_n(x)| \leq \omega_2(f;\tfrac{1}{k}) + k\Delta(f';\tfrac{1}{k})\lambda_n$$

for $0 \leq x \leq 1$.

This result can then be used [9] to obtain the lower estimate

$$\lambda_n \geq \frac{c}{n^2} ,$$

which is better than Newman's estimate [28],

$$\lambda_n \geq \frac{c}{n^2 \log n} \quad .$$

In fact, Freud has conjectured that it is very probable that $\lambda_n = O(n^{-2})$ holds. However, this remains to be shown.

Next we consider a function $f(x)$ defined on $[0,1]$ which is $(r-1)$-times continuously differentiable and for which $f^{(r-1)}(x)$ is an indefinite integral of a function $f^{[r]}(x)$ of bounded variation on $[0,1]$. We have already seen that Szüsz and Turán [29] and Freud [11]

have shown the existence of sequences $\{\lambda_{r,n}\}$ for which rational approximation to $f(x)$ yields the estimate

$$R_n(f;0,1) \le V(f^{[r]})\, \lambda_{r,n} ,$$

where $V(f^{[r]})$ denotes the total variation of $f^{[r]}(x)$ on $[0,1]$.

In this case the best known upper estimate of $\lambda_{r,n}$ is $O\left(\frac{\log^2 n}{n^{r+1}}\right)$ by Theorem 2.8, and in [10] G. Freud has shown the validity of the lower estimate,

$$\lambda_{r,n} \ge \frac{c}{n^{r+1}} .$$

Freud [11] has shown that a "piecewise smooth" function can be approximated by rational functions in the entire interval [a,b] with the same degree of approximation (up to a term of the order $O(e^{-c\sqrt{n}})$) just as it can be approximated by polynomials in finitely many subintervals. In addition, Freud conjectured that this result might still remain valid if the condition of polynomial approximation were replaced by rational approximation. This conjecture has now been proved by J. Szabados [25], and the following generalization of Theorem 2.8 is an application of his result.

THEOREM 2.10 [25]. Suppose $f(x)$ is a continuous function on $[0,1]$. Let

$$0 = x_1 < x_2 < \cdots < x_m < x_{m+1} = 1$$

be a subdivision of $[0,1]$ and assume that $f(x)$ is in the class $V^{(r)}$ $(r \ge 1)$ on $[x_k,x_{k+1}]$, $1 \le k \le m$. Then, for sufficiently large n, there exists a rational function $r_n(x)$ of order n such that

$$|f(x) - r_n(x)| \le c\, m^{r+1} \frac{\log^2 n}{n^{r+1}} + \exp\left(-\sqrt{\frac{n}{14m}}\right)$$

for $0 \le x \le 1$, where c is a positive constant (depending only on the maximum of the total variations of $f^{[r]}(x)$ in the subintervals $[x_k, x_{k+1}]$, $1 \le k \le m$).

Weighted Rational Approximation on an Infinite Interval

There has been limited interest shown in extending these results concerning rational approximation on a finite interval to weighted approximation by rational functions on an infinite interval. As a starting point to such investigations, G. Freud [8] used Newman's result directly to prove the existence of a rational function $r_n(x)$ of order n such that

$$||x| - r_n(x)| \le \frac{3}{2}(1 + x^2) \exp\left(-\sqrt{\frac{n-3}{2}}\right)$$

for $-\infty < x < \infty$. On the other hand, he showed that for every n there does not exist a rational function $r_n(x)$ of order n such that

$$||x| - r_n(x)| \le \frac{1}{4}(1 + x^2) \exp(-9\sqrt{n})$$

for $-\infty < x < \infty$.

It is known [31;p. 11] that if $f(x)$ is continuous on $(-\infty,\infty)$ and the limits

$$\lim_{x\to-\infty} f(x) \quad \text{and} \quad \lim_{x\to\infty} f(x)$$

exist, are equal and finite, then $f(x)$ can be uniformly approximated by rational functions on the whole real axis. However, G. Freud and J. Szabados [14] seem to have been the first to study the convergence

properties of this type of approximation. They obtained the following two results:

THEOREM 2.11 [14]. If the limits

$$\lim_{x\to-\infty} f(x) \quad \text{and} \quad \lim_{x\to\infty} f(x)$$

exist, are equal and finite, then for $n \geq 2$ there exists a rational function $r_n(x)$ of order n such that

$$|f(x) - r_n(x)| \leq 48\ \omega(\tfrac{1}{n})$$

for $-\infty < x < \infty$, where $\omega(\delta)$ is the modulus of continuity for the function $f(\tan \frac{x}{2})$ on $[-\pi,\pi]$.

We note that, if the limits

$$\lim_{x\to-\infty} f(x) \quad \text{and} \quad \lim_{x\to\infty} f(x)$$

are different, then weight functions must occur in the estimates for the degree of approximation.

THEOREM 2.12 [14]. Let $f(x)$ be a continuous function of bounded variation on $(-\infty,\infty)$. Then for $n \geq 1$ there exists a rational function $r_n(x)$ of order n such that

$$|f(x) - r_n(x)| \leq c\ (1 + x^2)\ \omega(\delta_{\left[\frac{n}{3}\right]})$$

for $-\infty < x < \infty$, where $\omega(\delta)$ is the modulus of continuity for the function $f\left(\frac{x}{1-|x|}\right)$ on $[-1,1]$ and δ_m is the unique solution of the equation

$$4 \log^2 \delta_m = m\,\omega(\delta_m)\ ,$$

$m = 0,1,2,\ldots.$

Other than the weight function $1 + x^2$, the estimate given in Theorem 2.12 is generally better than that given in Theorem 2.11.

Converse Theorems of Rational Approximation

Thus far we have only considered direct theorems about the convergence properties of sequences of rational functions. On the other hand, there exist converse theorems which assert that a function f has certain smoothness properties if $R_n(f)$ tends rapidly enough to zero.

In the case of polynomial approximation the Jackson-Bernstein theorems give us a way of characterizing, for example, functions which satisfy a Lipschitz condition of order α. However, the formulation of the converse theorems of rational approximation contains an exceptional set of arbitrarily small measure. Consequently, if $R_n(f;a,b)$ is very small, then we can only conclude that f is very smooth almost everywhere on the interval [a,b]. Newman's theorem on the rational approximation of $|x|$ shows that this is necessary.

The first results concerning converse theorems of rational approximation are due to A. A. Gončar [17]. E. P. Dolženko has obtained the following result:

THEOREM 2.13 [7]. If the function $f(x)$ is continuous on [a,b] and

$$\sum_{n=0}^{\infty} R_n(f) < \infty ,$$

then $f(x)$ is absolutely continuous on [a,b].

As a consequence of this theorem it follows that $f'(x)$ exists almost everywhere on [a,b], but no assumption about the order of convergence of $R_n(f)$ to zero ensures the existence of $f''(x)$ at even a single point in [a,b]. However, if we assume

$$\sum_{n=0}^{\infty} \left(\frac{R_n(f)}{n+1}\right)^{\frac{1}{r+1}} < \infty \qquad (r \geq 1) \quad ,$$

then the situation is different as is seen by the following theorem:

THEOREM 2.14 [6]. If the function $f(x)$ is continuous on [a,b] and

$$\sum_{n=0}^{\infty} \left(\frac{R_n(f)}{n+1}\right)^{\frac{1}{r+1}} < \infty \qquad (r \geq 1) \quad ,$$

then $f(x)$ is r-times continuously differentiable almost everywhere on [a,b].

It is also known [20;p.83] that if $r_n(x)$ is a rational function of order n such that $|r_n(x)| \leq M$ on [a,b], then $r_n(x)$ satisfies the Lipschitz condition of order 1 almost everywhere on [a,b]. This leads to the following result:

THEOREM 2.15 [6]. If the function $f(x)$ is continuous on [a,b] and for $0 < \alpha \leq 1$

$$R_n(f;a,b) \leq \frac{c}{n^{r+\alpha}} \qquad (r \geq 1) \quad ,$$

where c is a positive constant, then $f^{(r)}(x)$ exists everywhere on [a,b] except for a set E of arbitrarily small measure. Moreover, $f^{(r)}(x)$ satisfies a Lipschitz condition of order α on $[a,b] \sim E$.

A weaker form of this theorem had been formulated earlier by A. A. Gončar [17]. Under the same conditions for $f(x)$ in Theorem 2.15, he showed that $f(x)$ has an asymptotic r^{th} derivative almost everywhere on [a,b].

Other than allowing for exceptional sets of arbitrarily small measure, we observe that Theorem 2.15 is identical to the corresponding converse theorem [4;p.201] of Bernstein for approximation by polynomials. This analogy with Bernstein's theorem appears more vividly in the following result due to A. A. Gončar who extended the problem to functions defined on a finite perfect set.

THEOREM 2.16 [15]. Suppose the function $f(x)$ is continuous on $[a,b]$ and for $0 < \alpha \leq 1$, $\delta > 0$

$$R_n(f;a,b) \leq \frac{c}{n^{r+\alpha+\delta}} \qquad (r \geq 1) \quad ,$$

where c is a positive constant. Then, given $\varepsilon > 0$, there exists a perfect set $P_\varepsilon \subset [a,b]$ with the measure of $[a,b] \sim P_\varepsilon < \varepsilon$ such that the restriction of $f(x)$ to the set P_ε possesses an r^{th} derivative on P_ε. Moreover, $f^{(r)}(x)$ satisfies a Lipschitz condition of order α on this set P_ε.

These converse theorems are useful since they indicate those properties which are necessary for a function $f(x)$ in order to guarantee a given rate of convergence of the degree of approximation to $f(x)$ by rational functions. However, the problem of the exceptional sets still remains. P. Szüsz and P. Turán [27] raised the problem of characterization of functions without an exceptional set. They proposed to study the role of the minimum distance between any

point in [a,b] and the poles of the approximating rational function. In this connection J. Szabados [25] has obtained a number of results.

Let $z_1, z_2, \ldots, z_m$ be the poles of the rational function $r_n(x)$ of order n $(m \le n)$ and define

$$\delta(r_n) = \min \{ 1; \min_{1\le k\le m} \min_{-1\le x\le 1} |x - z_k| \} .$$

Let $p > 1$ and define

$$q = \begin{cases} 0 & \text{if } p \text{ is an integer} \\ [p] & \text{otherwise .} \end{cases}$$

The following theorem is typical of Szabados' results.

THEOREM 2.17 [25]. Let the function $f(x)$ be continuous on $[-1,1]$. If for $p > 1$ and $\varepsilon > 0$

$$R_n(f;-1,1) = O\left(\frac{\min \{ \delta(r_n); \min_{0\le j\le q} \delta(r_{[np]+j}) \}}{n^{1+\varepsilon}} \right) ,$$

then $f(x)$ is differentiable in $(-1,1)$.

In particular, if $r_n(x)$ is a polynomial, then $\delta(r_n) = 1$. In this case we see that the condition of Theorem 2.17 becomes

$$R_n(f;-1,1) = O\left(\frac{1}{n^{1+\varepsilon}}\right)$$

which corresponds to Bernstein's theorem, apart from the fact that Theorem 2.17 ensures that $f(x)$ is differentiable only in $(-1,1)$.

A Few Remarks on Rational Approximation in the Complex Plane

Rational functions can also be used successfully for the approximation of analytic and meromorphic functions of a complex variable, and in this section we shall briefly indicate some of the problems involved. In general, the theory of polynomial approximation to an analytic function relates the degree of approximation to regions of analyticity of the approximated function. The corresponding theory for approximation by rational functions has the same objective, but this has not yet been achieved.

There have been a number of results [35] concerning the rational function of best approximation to a complex valued function where the poles of the approximating rational function are known precisely or even asymptotically. However, the difficulty surrounding rational approximation results largely from a lack of knowledge of possible unprescribed poles of rational functions of best approximation. Best approximation here means minimizing the Chebyshev norm in the usual sense. If a function $f(z)$ is continuous on a closed bounded set E of the complex domain, then a rational function of best approximation to $f(z)$ on E exists (if E is dense in itself), but it need not be unique [35;p.356].

In the case of the real line, the results which we have already mentioned have involved rational functions where the poles are not fixed. Therefore the following question arises: <u>where should the poles of the approximating rational functions be placed in order to obtain the optimum degree of approximation</u>?

For example, the rational function constructed by D. J. Newman in Theorem 2.1 has the origin as the limit point of its poles. He showed that the choice of this particular rational function to approximate $|x|$ is nearly best possible, and his method was based on explicit formulae. In this connection J. L. Walsh has formulated the following general theorem:

THEOREM 2.18 [33]. If the function $f(z)$ can be approximated on a closed Jordan arc C with a degree of approximation $O(n^{-\alpha})$ $(\alpha > 0)$ by rational functions (of order n) whose poles have no limit point on C, then $f(z)$ can also be approximated on C with the same degree of approximation by polynomials (of degree n).

In particular, this theorem applies to every closed subinterval of [-1,1] which contains the origin. It shows that a rational function having <u>no</u> limit point of its poles on such a subinterval cannot converge to $|x|$ with a degree of approximation $O(n^{-\alpha})$ $(\alpha > 1)$ on the subinterval. Therefore Newman's rational function has at least one limit point of its poles on each closed subinterval of [-1,1] containing zero in its interior. From this we conclude that the origin must be the limit point of the poles.

CHAPTER III

INTERPOLATION BY RATIONAL FUNCTIONS

Introduction

A direct approach of approximating a function is to obtain an approximating function which assumes the same values as the function at a specified number of points in the domain of definition. This procedure is called interpolation, and the approximating function is said to interpolate the given function.

There are, of course, many choices for the points at which a function may be interpolated. The most natural choice is obviously to interpolate the given function at equidistant points.

Using methods similar to those of Newman, we shall construct a rational function of order n, which interpolates the function $f(x) = |x|$ at $2n + 1$ equidistant nodes in $[-1,1]$. Moreover, we shall show that this method of rational approximation to $|x|$ yields a degree of approximation equal to $O(n^{-1}\log^{-1} n)$. The function $|x|$ is important as a "test" function in the theory of approximation, and this importance has already been emphasized.

Finally, as an application of our result, we shall extend it to the case of a piecewise linear continuous function.

Interpolation to $|x|$ on Equidistant Nodes

Let

$$p_n(x) = \prod_{k=1}^{n} (x + \frac{k}{n}) ,$$

and define the rational function $r_n(x)$ by

$$r_n(x) = x \left(\frac{p_n(x) - p_n(-x)}{p_n(x) + p_n(-x)} \right) .$$

Clearly, $r_n(x)$ interpolates the function $f(x) = |x|$ at each point $\frac{k}{n}$, $k = 0, \pm 1, \pm 2, \ldots, \pm n$. Moreover, $r_n(x)$ is in the class $R(n,n)$ if n is even and $r_n(x)$ is in $R(n+1,n-1)$ if n is odd.

We shall prove the following result:

THEOREM 3.1. If $n \geq 7$, then

$$\big| |x| - r_n(x) \big| \leq \frac{3}{2\, n \log n}$$

for $-1 \leq x \leq 1$.

In order to prove this theorem we shall need two lemmas.

LEMMA 3.1. For $\frac{1}{n \log n} < x < \frac{1}{n}$, we have

$$\left| \frac{p_n(-x)}{p_n(x)} \right| < e^{-x\, n \log n} \qquad (n \geq 7) .$$

Proof. If $\psi(x) = 1 - x - e^{-x}$, then $\psi(0) = 0$ and $\psi'(x) \le 0$ for $x \ge 0$. It follows that $1 - x \le e^{-x}$ for $x \ge 0$. Therefore, for $\frac{1}{n \log n} < x < \frac{1}{n}$, we have

$$\left|\frac{p_n(-x)}{p_n(x)}\right| = \prod_{k=1}^{n} \frac{\frac{k}{n} - x}{\frac{k}{n} + x} = \prod_{k=1}^{n} \left(1 - \frac{2\,x\,n}{k + x\,n}\right)$$

$$\le \exp\left(- \sum_{k=1}^{n} \frac{2\,x\,n}{k + x\,n} \right) .$$

Now
$$\sum_{k=1}^{n} \frac{2\,x\,n}{k + x\,n} > 2\,x\,n \sum_{k=1}^{n} \frac{1}{k+1}$$

$$> 2\,x\,n\,\{\log\,(n+1) - 1\}$$

$$> x\,n\,\log n \qquad \text{if} \quad n \ge 7 .$$

Hence we have

$$\left|\frac{p_n(-x)}{p_n(x)}\right| < e^{-x\,n\,\log n} .$$

LEMMA 3.2. For $\frac{1}{n} < x < 1$, $x \ne \frac{k}{n}$, $1 \le k \le n$, we have

$$\left|\frac{p_n(-x)}{p_n(x)}\right| < \frac{1}{2\,n \log n} .$$

Proof. Suppose that $\frac{j}{n} < x < \frac{j+1}{n}$, $1 \le j \le n-1$.

Then we have

$$\left|\frac{p_n(-x)}{p_n(x)}\right| = \prod_{k=1}^{j} \frac{x - \frac{k}{n}}{x + \frac{k}{n}} \prod_{k=j+1}^{n} \frac{\frac{k}{n} - x}{\frac{k}{n} + x}$$

$$< \prod_{k=1}^{j} \frac{j+1-k}{j+k} \prod_{k=j+1}^{n} \frac{k-j}{k+j}$$

$$= \frac{j!\, j!\, (n-j)!}{(n+j)!}$$

$$= \left[\binom{n}{j}\binom{n+j}{j}\right]^{-1} .$$

Now $\binom{n}{j} \ge n$, $\binom{n+j}{j} \ge n+1$ for $1 \le j \le n-1$, so

$$\left|\frac{p_n(-x)}{p_n(x)}\right| < \frac{1}{n(n+1)} < \frac{1}{2n \log n} .$$

Proof of Theorem 3.1. Since $|x|$ and $r_n(x)$ are both even functions, it is sufficient to prove the required inequality for $0 \le x \le 1$.

For $0 \le x \le \frac{1}{n \log n}$, the inequality holds trivially, because here $p_n(x) > 0$, $p_n(-x) > 0$ and $p_n(x) \ge p_n(-x)$, so that we have $0 \le r_n(x) \le x$. Hence

$$\big||x| - r_n(x)\big| = x - r_n(x) \le x \le \frac{1}{n \log n} .$$

For $\frac{1}{n \log n} < x < \frac{1}{n}$, we have

$$|x - r_n(x)| = \left| x - x\left(\frac{p_n(x) - p_n(-x)}{p_n(x) + p_n(-x)}\right)\right|$$

$$= \frac{2x}{\left|\frac{p_n(x)}{p_n(-x)} + 1\right|}$$

$$\le \frac{2x}{\left|\frac{p_n(x)}{p_n(-x)}\right| - 1} .$$

Applying Lemma 3.1 we have

$$|x - r_n(x)| < \frac{2x}{e^{x n \log n} - 1}$$

$$= \frac{2x}{\frac{1}{2} e^{x n \log n} + \left(\frac{1}{2} e^{x n \log n} - 1\right)} .$$

Now $x > \frac{1}{n \log n}$ implies that $\frac{1}{2} e^{x n \log n} > \frac{1}{2} e > 1$.

Therefore we have

$$|x - r_n(x)| < 4 x e^{-x n \log n} .$$

Next we let

$$\phi(x) = 4 x e^{-x n \log n} .$$

Then $$\phi'(x) = 4\,e^{-x\,n\log n} - 4\,xn\,\log n\,e^{-x\,n\log n} ,$$

so that $\phi'(x) = 0$ if and only if $x = \frac{1}{n\,\log\,n}$. Since $\phi(x) \geq 0$ on $[0,\frac{1}{n}]$ with $\phi(0) = 0$ and $\phi(\frac{1}{n}) = \frac{1}{n^2}$, it follows that $\phi(x)$ achieves its maximum at $x = \frac{1}{n\,\log\,n}$, whose value is $\frac{4\,e^{-1}}{n\,\log\,n}$.

Hence we have

$$|x - r_n(x)| < \frac{4\,e^{-1}}{n\,\log\,n} < \frac{3}{2\,n\,\log\,n}$$

for $0 \leq x < \frac{1}{n}$.

Since $r_n(x)$ interpolates at $x = \frac{k}{n}$, $1 \leq k \leq n$, we have only to consider $\frac{1}{n} < x < 1$, $x \neq \frac{k}{n}$, $1 \leq k \leq n$. Now we apply Lemma 3.2 to obtain

$$\begin{aligned} |x - r_n(x)| &\leq \frac{2\,x}{\left|\frac{p_n(x)}{p_n(-x)}\right| - 1} \\ &< \frac{2}{2\,n\,\log\,n - 1} \\ &< \frac{3}{2\,n\,\log\,n} . \end{aligned}$$

This completes the proof.

We remark that our estimate of $O(n^{-1}\log^{-1}n)$ in Theorem 3.1 is the best possible for the rational function which we have constructed. This is readily seen upon putting $x = \frac{1}{n\,\log\,n}$. In this case we have

$$|x - r_n(x)| = \frac{1}{n \log n} \frac{2}{\frac{p_n(x)}{p_n(-x)} + 1} .$$

Now

$$\frac{p_n(x)}{p_n(-x)} = \prod_{k=1}^{n} \frac{\frac{k}{n} + \frac{1}{n \log n}}{\frac{k}{n} - \frac{1}{n \log n}} = \prod_{k=1}^{n} \left(1 + \frac{2}{k \log n - 1}\right)$$

$$\leq \prod_{k=1}^{n} \left(1 + \frac{c_1}{k \log n}\right) \leq \exp\left(\frac{c_1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\right)$$

for some positive constant c_1. But

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k} < 1 + \int_1^n \frac{dt}{t} < 2 \log n .$$

Therefore $\frac{p_n(x)}{p_n(-x)} \leq c_2$ for some positive constant c_2. It follows that

$$||x| - r_n(x)| \geq \frac{c}{n \log n}$$

for $-1 \leq x \leq 1$, where c is a positive constant.

Interpolation to a Piecewise Linear Continuous Function

The result of Theorem 3.1 can be extended to the case of a continuous function $f(x)$ which is piecewise linear. If such an $f(x)$ is linear on m subintervals of equal length in $[-1,1]$, then a rational function of order n can be constructed which interpolates $f(x)$ at approximately $[\frac{n}{m}]$ equally spaced points, including the $m + 1$ points of subdivision. We shall prove the following result:

THEOREM 3.2. Let $x_0, x_1, \ldots, x_m$ be $m + 1$ equally spaced points in $[-1,1]$. Suppose $f(x)$ is a continuous function which is linear on each subinterval $[x_{\nu-1}, x_\nu]$, $1 \leq \nu \leq m$. Then, for $n \geq m^2 - m + 1$, there exists a rational function $R_n(x)$ of order n such that

$$|f(x) - R_n(x)| \leq \frac{c}{n \log n}$$

for $-1 \leq x \leq 1$, where c is a positive constant (depending only on f). Moreover, $R_n(x)$ interpolates $f(x)$ at $m\left[\frac{n-1}{m(m-1)}\right]$ equally spaced points in $[-1,1]$, including the points $x_0, x_1, \ldots, x_m$.

Before we prove this theorem we shall need two lemmas, including a special representation for $f(x)$.

LEMMA 3.3. The function $f(x)$ can be represented by

$$f(x) = a_0 + a_m x + \sum_{\nu=1}^{m-1} a_\nu |x - x_\nu|$$

where the coefficients a_ν, $0 \leq \nu \leq m$, depend only on f.

<u>Proof</u>. We can represent $f(x)$ by

$$f(x) = \sum_{\nu=0}^{m-1} b_\nu \, M(x;x_\nu)$$

where $M(x;-1)$ is linear and coincides with $f(x)$ on $[-1,x_1]$, and for $1 \leq \nu \leq m - 1$

$$M(x;x_\nu) = \frac{(x - x_\nu) + |x - x_\nu|}{2},$$

$-1 \leq x \leq 1$. (The coefficients b_ν can be determined successively).

It follows that we can represent $f(x)$ by

$$f(x) = a_0 + a_m x + \sum_{\nu=1}^{m-1} a_\nu |x - x_\nu|$$

with suitable coefficients a_ν, $0 \le \nu \le m$, depending only on f.

LEMMA 3.4. For $-1 \le x \le 1$, $0 \le \nu \le m$, we have

$$\left| |x - x_\nu| - r_N(x - x_\nu) \right| \le \frac{c_1}{N \log N} \qquad (N \ge 2) \quad ,$$

where c_1 is a positive constant.

Proof. Applying Theorem 3.1 we have

$$\left| |x| - r_N(x) \right| \le \frac{c_2}{N \log N}$$

for $-1 \le x \le 1$.

Suppose $1 < x \le 2$. Then

$$\left| \frac{p_N(-x)}{p_N(x)} \right| = \prod_{k=1}^{N} \frac{x - \frac{k}{N}}{x + \frac{k}{N}} = \prod_{k=1}^{N} \left(1 - \frac{2k}{k + Nx} \right)$$

$$\le \exp\left(-2 \sum_{k=1}^{N} \frac{k}{k + Nx} \right) \quad .$$

Now $$\sum_{k=1}^{N} \frac{k}{k + Nx} \ge \sum_{k=1}^{N} \frac{k}{k + 2N} > \int_0^N \frac{t\,dt}{t + 2N}$$

$$= N + 2N \log\frac{2}{3} > \frac{1}{6} N \quad .$$

Therefore we have

$$\left|\frac{p_N(-x)}{p_N(x)}\right| < e^{-\frac{1}{3}N} \le \frac{c_3}{N \log N}$$

for $1 < x \le 2$. Hence

$$|x - r_N(x)| \le \frac{2x}{\left|\frac{p_N(x)}{p_N(-x)}\right| - 1} \le \frac{c_4}{N \log N}$$

for $1 < x \le 2$.

Similarly, for $-2 \le x < -1$, we have

$$\left|\frac{p_N(x)}{p_N(-x)}\right| \le \frac{c_3}{N \log N} \quad ,$$

and it follows that

$$\big||x| - r_N(x)\big| \le \frac{c_1}{N \log N}$$

throughout $[-2,2]$, where c_1 is a positive constant. Hence we have

$$\big||x - x_\nu| - r_N(x - x_\nu)\big| \le \frac{c_1}{N \log N}$$

for $-1 \le x \le 1$, $0 \le \nu \le m$.

<u>Proof of Theorem 3.2</u>. By Lemma 3.3 we have

$$f(x) = a_0 + a_m x + \sum_{\nu=1}^{m-1} a_\nu |x - x_\nu|$$

with suitable coefficients a_ν, $0 \le \nu \le m$, which depend only on f.

Let $N = m\left[\frac{n-1}{m(m-1)}\right]$ and define the rational function $R(x)$ of order $(m-1)N+1$ by

$$R(x) = a_0 + a_m x + \sum_{\nu=1}^{m-1} a_\nu\, r_N(x - x_\nu)$$

where

$$r_N(x) = x\left(\frac{p_N(x) - p_N(-x)}{p_N(x) + p_N(-x)}\right)$$

and

$$p_N(x) = \prod_{k=1}^{N}(x + \tfrac{k}{N}) \quad .$$

Clearly, $R(x)$ interpolates $f(x)$ at each point $\frac{k}{N}$, $0 \le k \le N$.

Moreover, by Lemma 3.4 we have

$$|f(x) - R(x)| \le \sum_{\nu=1}^{m-1} |a_\nu|\,\big||x - x_\nu| - r_N(x - x_\nu)\big|$$

$$\le \frac{c}{N \log N}$$

for $-1 \le x \le 1$, where c is a positive constant depending only on f.

Now $R(x)$ is of order no greater than n since

$$(m-1)\,N + 1 \le (m-1)\, m\, \frac{n-1}{m(m-1)} + 1 = n \quad .$$

Letting $R_n(x) = R(x)$, we have the required result. This completes the proof.

Because of the remark following Theorem 3.1, it is clear that the estimate $O(n^{-1}\log^{-1} n)$ in Theorem 3.2 cannot be improved.

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